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An Invariant Determining the Witt Class of a Unitary Transformation over a Semisimple Ring*

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1. INTRODUCTION

The first Witt group, $W_1{}^\lambda(A)$, of stable unitary transformations, $U^\lambda(A)$, of an even λ-symmetric hyperbolic form over a ring-with-involution A ($\lambda = \pm 1$) is defined in [2, I]. The main theorem (3.1) in this paper shows that if $\sigma \in U_{2n}^\lambda(A)$ is written

$$\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{1.1}$$

with $(n \times n)$-matrices as entries, then the class of σ in $W_1{}^\lambda(A)$ is determined by the cokernel of α, viewed as a map $A^n \to A^n$. In particular, the class of σ is zero if and only if cok α supports a nonsingular even $(-\lambda)$-hermitian form.

By definition $W_1{}^\lambda(A)$ is the commutator quotient of $U_{2n}^\lambda(A)$ (n large), modulo matrices of the form

$$H(a) = \begin{pmatrix} a & 0 \\ 0 & \bar{a}^{-1} \end{pmatrix} \in U_{2n}^\lambda(A),$$

where $a \in GL_n(A)$. This is explained further and other definitions and results from unitary algebraic K-theory are recalled in Section 2, the main source for which is the excellent foundational paper of Bass [2]. In Section 3 we state and prove the main result (3.1) and in (3 14) extend it to a larger (probably largest possible) class of rings. An interesting consequence of (3.1) is (3 12), where the constructions in (3.1) are used to decompose the matrix σ of (1.1) into simpler ones from knowledge of α and γ. In Section 4 we make fairly complete computations. Many of these results are to be found in [4] or [16]. However, these treatments involve a variety of techniques (e g , Clifford algebras and Morita theory) whereas (3.1) and a theorem of Bass ((4 2), proved independently by the

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author)—both essentially exercises in linear algebra—yield a fairly complete and unified approach. Finally, in Section 5 we sketch connections with a so-called Rothenberg sequence and with the topological motivation for (3.1).

Throughout this paper it will be assumed that the reader is familiar with the theory of semisimple rings, as exposed for example in [5]; explicit references are often omitted. It should also be noted that we deal here with *split* or *even forms only* (see (2 2)(5)).

I would like to acknowledge helpful conversations with Hyman Bass and Goro Shimura.

2. Notation and Review of Basic Results and Definitions

(2.1) In general we will use the results and notational conventions of [2] Let (A, λ, Λ) be a unitary ring in the sense of [2, Chap. 1, 4.1], where, for simplicity of exposition, we require special values for λ and Λ. Thus, A is a ring with involution, $a \to \bar{a}$, $\lambda = \pm 1$, and $\Lambda^\lambda = \{a - \lambda\bar{a} \mid a \in A\}$. If α is an $n \times n$ matrix over A we write $\bar{\alpha}$ for its conjugate transpose. Denote by $\Lambda_n{}^\lambda$ the set of all α such that $\alpha = -\lambda\bar{\alpha}$ and the diagonal entries of α lie in Λ^λ. All modules M over A will be *right* A-modules.

If $\overline{M}$ denotes $\mathrm{Hom}_A(M, A)$, then $\overline{M}$ can be given a right A-module structure using the involution on A. A map $f: F \to G$ between free A-modules with chosen bases will be identified with its matrix α; conversely, any matrix α will be identified with a map of free modules Finally, the induced map $\bar{f}$: $\bar{G} \to \bar{F}$ has matrix $\bar{\alpha}$

(2.2) If M and N are A-modules, a function g: $M \times N \to A$ is called a *sesquilinear pairing* if

(1) $g(ma, nb) = \bar{a}g(m, n)b$,

(2) $g(m, n + n') = g(m, n) + g(m, n')$

(3) $g(m + m', n) = g(m, n) + g(m', n)$

for all $m, m' \in M$; $n, n' \in N$; $a, b \in A$.

It is called *nonsingular* if the adjoint maps induced by g [2, I 2.2]

$$N \to \overline{M} \qquad \text{and} \qquad M \to \overline{N}$$

are isomorphisms.

If g satisfies (1)–(3) above and $N = M$, g: $M \times M \to A$ is said to be λ-hermitian provided

(4) $g(m, m') = \lambda\overline{g(m', m)}$,

(5) $g(m, m) \in \Lambda^{-\lambda}$

for all $m, m' \in M$ If, in addition, g is nonsingular, M is said to *support a λ-form.* Let $M = \bar{N} + N$ The *hyperbolic* form $h\colon M \times M \to A$ is the λ-hermitian form given by $h \mid N \times N \equiv 0 \equiv h \mid \bar{N} \times \bar{N}$ and $h \mid \bar{N} \times N$ is the natural sesquilinear pairing $\mathrm{Hom}(N, A) \times N \to A$.

(2.3) Define $KU_0{}^\lambda(A)$ ($= KU_0{}^\lambda(A, A)$ in [2]) to be the Grothendieck group on isomorphism class of nonsingular λ-hermitian forms on projective A-modules. There is a map $H_0 : K_0(A) \to KU_0{}^\lambda(A)$ sending a projective module N to the hyperbolic form h defined on $\bar{N} + N$ (2.2); and there is an obvious forgetful map $F_0 : KU_0{}^\lambda(A) \to K_0(A)$. We define $W_0{}^\lambda(A)$ to be the cokernel of H_0.

(2.4) The set of isometries of the hyperbolic form h on $\bar{A}^n + A^n \cong A^{2n}$ form a group $U_{2n}^\lambda(A)$ whose elements σ may be written

$$\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL_{2n}(A), \tag{2.4a}$$

where $\alpha, \beta, \gamma, \delta$ are $n \times n$ matrices satisfying the conditions [2, II.4.1.2]

(i) $\alpha\bar{\delta} + \lambda\beta\bar{\gamma} = I$,

(ii) $\beta\bar{\alpha}, \gamma\bar{\delta}, \bar{\beta}\delta, \bar{\gamma}\alpha \in A_n{}^\lambda$

(see (2.1) for the definition of $A_n{}^\lambda$; there is a mistake in sign [2, II.4.1.]). The matrices α and γ define a map $(\gamma, \alpha)\colon A^n \to \bar{A}^n + A^n$ which is clearly injective; further, the relations in (ii) above easily imply that h annihilates the image of (γ, α). Thus, if $x, y \in A^n$ (since $h(\alpha(x), \alpha(y)) = 0 = h(\gamma(x), \gamma(y))$ by definition of h)

$$h(\alpha(x), \gamma(y)) + h(\gamma(x), \alpha(y)) = 0.$$

The reader may verify that $[x, y] = h(\alpha(x), \gamma(y))$ defines a $(-\lambda)$-hermitian form on A^n whose matrix is $\bar{\gamma}\alpha$ (see (2.4) (ii)).

Conversely, given a homomorphism (γ, α). $G \to \bar{F} + F$, where F and G are A-free on n generators, such that $\mathrm{im}(\gamma, \alpha)$ is self-annihilating under the natural hyperbolic form on $\bar{F} + F$ (2 2), we can find a corresponding element $\sigma \in U_{2n}^\lambda$ (2 4a). This is proved in [7; 17, Chap. 6], where $\mathrm{im}(\gamma, \alpha)$ is called a *subkernel* (of the hyperbolic structure on $\bar{F} + F$). It is further shown in [7, 12] that one may define an equivalence relation on "subkernels" leading to a one-to-one correspondence between the resultant equivalence classes and the group $W_1{}^\lambda(A)$. Thus we may think of the elements of $W_1{}^\lambda(A)$ as pairs $\bar{F}$ and G of subkernels of $\bar{F} + F$.

If $\sigma \in U_{2n}^\lambda(A)$ is as above, and $\sigma' = \left(\begin{smallmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{smallmatrix}\right) \in U_{2m}^\lambda(A)$, set

$$\sigma \perp \sigma' = \begin{pmatrix} \alpha & 0 & \beta & 0 \\ 0 & \alpha' & 0 & \beta' \\ \gamma & 0 & \delta & 0 \\ 0 & \gamma' & 0 & \delta' \end{pmatrix} \in U_{2(m+n)}^\lambda(A).$$

Letting $I_2 = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$ and $n \to \infty$ we obtain $U^\lambda(A)$ using the sequence of stabilizations $\sigma \to \sigma \perp I_2$, defining functions $U_{2n}^\lambda(A) \to U_{2(n+1)}^\lambda(A)$ for each n.

(2.5) Here are some special elements of the unitary group $U_{2n}^\lambda(A)$:

(i) $w_1 = \begin{pmatrix}0 & 1\\ \lambda & 0\end{pmatrix} \in U_2{}^\lambda(A)$; $w_n = w_{n-1} \perp w_1 \in U_{2n}^\lambda(A)$.

(ii) There is a homomorphism $H\colon GL_n(A) \to U_{2n}^\lambda(A)$ sending $a \in GL_n(A)$ to

$$H(a) = \begin{pmatrix}a & 0\\ 0 & \bar{a}^{-1}\end{pmatrix}.$$

This stabilizes to a homomorphism $H\colon GL(A) \to U^\lambda(A)$ whose abelianization is a homomorphism $H_1 : K_1(A) \to KU_1{}^\lambda(A)$. The cokernel of H_1 is called the Witt group, $W_1{}^\lambda(A)$.

We define the Wall group $L_\lambda(A)$ to be the quotient $KU^\lambda(A)/\langle H_1(K_1(A)), w_1\rangle$, where if X is a subset of a group then $\langle X\rangle$ is the subgroup generated by X.

(iii) Let $\rho, \tau \in \Lambda_n{}^\lambda$ and set $X_+(\rho) = \left(\begin{smallmatrix}I & \rho\\ 0 & I\end{smallmatrix}\right)$, $X_-(\tau) = \left(\begin{smallmatrix}I & 0\\ \tau & I\end{smallmatrix}\right) \in U_{2n}^\lambda(A)$.

In [2] it is shown that $\langle H(a), X_+(\rho), X_-(\tau)\rangle$ (for $a \in GL(A)$, $\rho, \tau \in \Lambda_n{}^\lambda$) is a subgroup of $U^\lambda(A)$ containing the commutator subgroup and

$$U^\lambda(A)/\langle H(a), X_+(\rho), X_-(\tau)\rangle = W_1{}^\lambda(A).$$

(2.6) Let $[A]$ denote the class of the module A in $K_0(A)$ and set $\tilde{K}_0(A) = K_0(A)/\langle [A]\rangle$. There is a natural Z_2-action on $K_0(A)$ and $\tilde{K}_0(A)$ given by $P \to \bar{P}$, for projective P. Thus we can form the (Tate) cohomology groups (which are of exponent 2)

$$\hat{H}^n(Z_2 ; K_0(A)) = \frac{\{x \in K_0(A) \mid x = (-1)^n \bar{x}\}}{\{y \in K_0(A) \mid y = z + (-1)^n \bar{z}\}}.$$

Similarly we can form $\hat{H}^n(Z_2 ; \tilde{K}_0(A))$.

3. The Main Result

Our main result is that an element $\sigma = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in U_{2n}^\lambda(A)$ represents zero in $W_1{}^\lambda(A)$ ($L_\lambda(A)$) if and only if cok α (stably) supports a $(-\lambda)$-form (2.2),

$$g\colon \operatorname{cok} \alpha \times \operatorname{cok} \alpha \to A,$$

where we view α as a homomorphism $A^n \to A^n$. To state the result precisely, note that the map F_0 of (2.3) induces homomorphisms

$$KU_0^\lambda(A) \to \hat{H}^0(Z_2 ; K_0(A))$$

and (3.0)

$$KU_0^\lambda(A) \to \hat{H}^0(Z_2 ; \tilde{K}_0(A))$$

whose cokernels we denote $\Phi^\lambda(A)$ and $\tilde{\Phi}^\lambda(A)$.

(3.1) THEOREM. *Let A be a semisimple ring with involution. Then there are natural isomorphisms*

$$f\colon W_1^\lambda(A) \overset{\cong}{\to} \Phi^{(-\lambda)}(A),$$

$$\tilde{f}.L_\lambda(A) \overset{\cong}{\to} \tilde{\Phi}^{(-\lambda)}(A).$$

In each case the map is given on representatives by sending $\sigma = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in U_{2n}^\lambda(A)$ to cok α.

(3 2) CONVENTION. To shorten notation in this proof we always take $\sigma = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in U_{2n}(A)$. The matrices α, β, γ, and δ are identified with homomorphisms of free modules.

Proof of (3.1) We will deal with the map f only and derive the result for $\tilde{f}$ The proof is divided into four steps

Step 1. We show f takes values in $\Phi^{(-\lambda)}(A)$ by showing that for $\sigma \in U_{2n}^\lambda(A)$, cok $\alpha \cong \overline{\text{cok }\alpha}$. For purposes of identification we take the map (γ, α) of (2.4) to map G to $\bar{F} + F$, where F and G are isomorphic to A^n. From the exact sequence ker $\alpha \to G \to F \to$ cok α and the fact that A is semisimple, it suffices to construct a nonsingular (sesquilinear) pairing g: ker $\alpha \times$ cok $\alpha \to A$. Let $\bar{g}$: ker $\alpha \times F \to A$ be given by

$$\bar{g}(x, f) = h(\gamma(x), f),$$

for $x \in \ker \alpha, f \in F$, and where h is as in (2.4) Since $h(\alpha(x), \gamma(y)) + h(\gamma(x), \alpha(y)) = 0$ for all $x, y \in G$ (2.4), $\bar{g}(x, f) = 0$ if $x \in \ker \alpha$, $f \in \operatorname{Im} \alpha$ Hence we obtain g: ker $\alpha \times$ cok $\alpha \to A$. Given $x \in \ker \alpha$, $\bar{g}(x, f)$ cannot be zero for all $f \in F$ since $\gamma(x) \neq 0$ ((γ, α) is injective) and since h would then be singular. Hence for dimension reasons (A is semisimple) g is nonsingular. This completes the proof that cok $\alpha \cong \overline{\text{cok }\alpha}$.

Step 2. f is well defined Given $\sigma \in U_{2n}^\lambda(A)$, we must show that [cok α] $\in \Phi$ is unaffected by multiplication of σ on the left or right by elements of type

((2.5)(ii), (iii)) $H(\alpha)$, $X_+(\rho)$, $X_-(\tau)$. For $H(a)$ this is obvious. For $X_+(\rho)$ and $X_-(\tau)$ we only have to show $f(\sigma X_-(\tau)) = f(\sigma) = f(X_+(\rho)\sigma)$, where

$$X_+(\rho)\,\sigma = \begin{pmatrix} \alpha + \rho\gamma & \beta + \rho\delta \\ \gamma & \delta \end{pmatrix} \quad \text{and} \quad \sigma X_-(\tau) = \begin{pmatrix} \alpha + \beta\tau & \beta \\ \gamma + \delta\tau & \delta \end{pmatrix}.$$

For this it clearly suffices to prove the following lemma.

(3.3) LEMMA $f\begin{pmatrix} \gamma & \delta \\ \alpha & \beta \end{pmatrix} = f\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = f\begin{pmatrix} \beta & \alpha \\ \delta & \gamma \end{pmatrix}$.

Proof. We prove the first equality and leave the second to the reader. We have $\alpha\colon G \to F$ and $\gamma\colon G \to \bar{F}$ as above. From the composition $\bar{\gamma}\alpha\colon G \to F \to \bar{F}$ we obtain the exact sequence

$$\ker \alpha \longrightarrow \ker \bar{\gamma}\alpha \longrightarrow \ker \bar{\gamma} \xrightarrow{d} \operatorname{cok} \alpha \longrightarrow \operatorname{cok} \bar{\gamma}\alpha \longrightarrow \operatorname{cok} \bar{\gamma}. \tag{3.4}$$

From (2.4)(i) we have $\alpha\bar{\delta} + \lambda\beta\bar{\gamma} = I_n$ so that $\ker \bar{\gamma} \subseteq \operatorname{im} \alpha$. By construction of (3.4), $d = 0$. Hence $\operatorname{cok} \bar{\gamma}\alpha \cong \operatorname{cok} \alpha + \operatorname{cok} \bar{\gamma}$. But since $\bar{\gamma}\alpha \in \Lambda_n^{\lambda}$ ((2.4)(ii)), $\operatorname{im}(\bar{\gamma}\alpha)$ supports a form, so that from $\operatorname{im}(\bar{\gamma}\alpha) \to \bar{F} \to \operatorname{cok}(\bar{\gamma}\alpha)$ we obtain $[\operatorname{cok}(\bar{\gamma}\alpha)] = 0$ in $\Phi^{(-\lambda)}(A)$ (see Lemma (3 7)). Hence, since $\hat{H}^0$ is of exponent 2, $[\operatorname{cok} \alpha] = [\operatorname{cok} \bar{\gamma}]$ in $\Phi^{(-\lambda)}(A)$. The isomorphisms $\operatorname{cok} \bar{\gamma} \cong \overline{\ker \gamma}$ and $\ker \gamma \cong \operatorname{cok} \gamma$ are easily verified, so in $\Phi^{(-\lambda)}(A)$,

$$[\operatorname{cok} \alpha] = [\operatorname{cok} \bar{\gamma}] = [\overline{\ker \gamma}] = [\overline{\operatorname{cok} \gamma}].$$

But in Step 1 we showed that $\operatorname{cok} \alpha \cong \overline{\operatorname{cok} \alpha}$ and from this it is clear that $[\operatorname{cok} \alpha] = [\operatorname{cok} \gamma]$ in $\Phi^{(-\lambda)}(A)$. This completes the proof of (3.3).

To complete Step 2 we show f is a homomorphism. For elements $\sigma, \tau \in U_{2n}^{\lambda}(A)$ we have the easily verifiable identity in $U_{4n}^{\lambda}(A)$:

$$\sigma\tau \perp I_{2n} = (\sigma\tau \perp (\sigma\tau)^{-1})(\sigma^{-1} \perp \sigma)(\sigma \perp \tau) \tag{3.5a}$$

But since $\sigma\tau \perp (\sigma\tau)^{-1}$ and $\sigma^{-1} \perp \sigma$ are products of matrices of the type given in (2.5(ii), (iii)) above [2, II.3.7.1], and since (obviously) $f(\sigma \perp \tau) = f(\sigma) + f(\tau)$, we have

$$f(\sigma\tau) = f(\sigma\tau \perp I_{2n}) = f[(\sigma\tau \perp (\sigma\tau)^{-1})(\sigma^{-1} \perp \sigma)(\sigma \perp \tau)] = f(\sigma \perp \tau) \tag{3 5b}$$

(by the first part of Step 2) $= f(\sigma) + f(\tau)$. A standard stabilization argument, using $f(\sigma) = f(\sigma \perp I_{2m})$, for all m, (used in (3.5b)) shows f is a well-defined homomorphism as required.

Step 3. f is a monomorphism. Suppose $\sigma \in U_{2n}^{\lambda}(A)$ is such that $f(\sigma) = 0$. We may assume n is even by replacing σ by $\sigma \perp I_2$, if necessary. If $n = 2k$, $w_n = w_k \perp w_k$ and since $(w_k)^2 = H(\lambda I_k)$, Eq. (3.5a) shows $w_n = 0$ in $W_1^{\lambda}(A)$ (taking $\sigma = \tau = w_k$) Hence $f(w_n\sigma) = f(\sigma) = 0$, so $[\operatorname{cok} \gamma] = 0$ in $\Phi^{(-\lambda)}(A)$.

This means that in $K_0(A)$, we have

$$[\operatorname{cok}\gamma] = ([N+\bar{N}] - [P+\bar{P}]) + ([Q]-[R]),$$

where N and P are projective, and Q and R support nonsingular $(-\lambda)$-forms (see (2 2). Hence, since A is semisimple there is an isomorphism of A-modules

$$\operatorname{cok}\gamma + P + \bar{P} + R \cong N + \bar{N} + Q. \tag{3.6a}$$

By adding $\bar{Q}$ (which supports a $(-\lambda)$-form) and the inverses (in $\tilde{K}_0$) of N and $\bar{N}$ to both sides of (3.6a); and then the inverses of Q and $\bar{Q}$ to both sides, we find that

$$\operatorname{cok}\gamma + S \cong A^p, \tag{3 6b}$$

where S supports a $(-\lambda)$-form and p is even

(3 7) LEMMA. *Let U and V be A-modules such that $U + V \cong A^p$. Then if U supports a λ-form, $V + A^p$ supports a λ-form.*

Proof Imbed the form on U into the hyperbolic form on $\bar{U} + \bar{V} + U + V$ [2, I 3 6.(c)] and take the orthogonal complement. This gives the form on $V + \bar{U} + \bar{V} \cong V + A^p$. Details are left to the reader.

Now Lemma (3.7) applied to (3.6b) allows us to conclude that $\operatorname{cok}\gamma + A^p$ supports a form for some even p. From the relation

$$\operatorname{cok}\gamma + A^p \cong \operatorname{cok}(\gamma \perp w_p)$$

and Step 2 (where we noted $w_p = 0$ in $W_1^\lambda(A)$ if p is even), we may assume $\operatorname{cok}\gamma$ supports a form. The exact sequence

$$\ker\gamma \longrightarrow G \xrightarrow{\gamma} \bar{F} \longrightarrow \operatorname{cok}\gamma$$

shows that $\bar{F} + \ker\gamma$ supports a form so that, stabilizing σ again, we may assume $\ker\gamma$ supports a form Since $(\gamma, \alpha)\colon G \to \bar{F} + F$ is a monomorphism (2 4) $\ker\alpha \cap \ker\gamma = (0)$. Hence $\alpha\,(\ker\gamma) \subset F$ supports a form We will complete Step 3 by showing that this fact about $\alpha\,(\ker\gamma)$ allows us to find a map $\tau\colon F \to \bar{F}$ which is the adjoint of a $(-\lambda)$-hermitian form $\psi\colon F \times F \to A$ (so that the matrix of τ is in Λ_n^λ (2.1)) and with $\gamma + \tau\alpha\colon G \to \bar{F}$ an isomorphism. One may now easily verify that

$$X_-(-\lambda\bar{B}\alpha)\, H(B^{-1})\, w_n X_-(\tau)\sigma = X_+(B^{-1}C), \tag{3.8}$$

where $B = \gamma + \tau\alpha$, and $C = \delta + \tau\beta$.

Hence it remains to construct $\tau\colon F \to \bar{F}$. Since $\ker\alpha \cap \ker\gamma = (0)$, $Y \subseteq G$ may be chosen so that $\ker\alpha + \ker\gamma + Y = G$. Let $\gamma\,(\ker\alpha)$ and $\alpha\,(\ker\gamma)$ have complements chosen in $\bar{F}$ and F, respectively, so that we may think of them as summands. With these choices, $\overline{\gamma(\ker\alpha)}$ and $\overline{\alpha(\ker\gamma)}$ become summands of F and $\bar{F}$. Since $h(\alpha(x), \gamma(y)) + h(\gamma(x), \alpha(y)) = 0$ for all $x, y \in G$ (2.4), $h(\gamma(\ker\alpha), \alpha(\ker\gamma)) \equiv 0$. Since $h \mid \bar{F} \times F$ is the natural pairing, we must have $\overline{\gamma(\ker\alpha)} \cap \alpha(\ker\gamma) = (0)$ in F and $\overline{\alpha(\ker\gamma)} \cap \gamma(\ker\alpha) = (0)$ in $\bar{F}$. We claim

$$\begin{aligned} &(\mathrm{i}) \quad \alpha(Y) \cap (\overline{\gamma(\ker\alpha)} + \alpha(\ker\gamma)) = (0), \\ &(\mathrm{ii}) \quad \gamma(Y) \cap (\gamma(\ker\alpha) + \overline{\alpha(\ker\gamma)}) = (0). \end{aligned} \tag{3.9}$$

The proof of (i) will give that of (ii). So let $\alpha(y) = u + v$, where $u \in \overline{\gamma(\ker\alpha)}$, $v \in \alpha(\ker\gamma)$, $y \in Y$. Then if $x \in \ker\alpha$, $h(\gamma(x), \alpha(y)) = 0$ (recall $h(\alpha(x), \gamma(y)) + h(\gamma(x), \alpha(y)) = 0$, for all $x, y \in G$); also $h(\gamma(x), v) = 0$ because $h(\gamma(\ker\alpha), \alpha(\ker\gamma)) \equiv 0$, as noted above. Hence we have

$$h(\gamma(x), u) = h(\gamma(x), \alpha(y)) - h(\gamma(x), v) = 0 - 0 = 0.$$

But this could not hold for all $x \in \ker\alpha$, since $u \in \overline{\gamma(\ker\alpha)}$, and $h \mid \gamma(\ker\alpha) \times \overline{\gamma(\ker\alpha)}$ is nonsingular. This proves (3.9)

From (3.9) we have isomorphisms

$$\begin{aligned} \alpha(Y) + \overline{\gamma(\ker\alpha)} + \alpha(\ker\gamma) &\cong F, \\ \gamma(Y) + \gamma(\ker\alpha) + \overline{\alpha(\ker\gamma)} &\cong \bar{F}. \end{aligned} \tag{3.10}$$

Now define $\tau: F \to \bar{F}$ to be zero on the summand $\alpha(Y) + \overline{\gamma(\ker\alpha)}$ and to map $\alpha(\ker\gamma) \overset{\cong}{\to} \overline{\alpha(\ker\gamma)} \subseteq \bar{F}$ by the adjoint of a nonsingular $(-\lambda)$-hermitian form which we have shown above to exist. Since τ annihilates $\alpha(Y)$ and maps $\alpha(\ker\gamma)$ onto a summand $\overline{\alpha(\ker\gamma)}$ disjoint from $\gamma(\ker\alpha) + \gamma(Y) = \gamma(\ker\alpha + Y) = \gamma(G)$, $\gamma + \tau\alpha$ must be surjective, hence an isomorphism. This completes the proof of Step 3

Step 4. f is surjective. Suppose given an A-module M, and an isomorphism $M \cong \overline{M}$ By (2.4), it suffices to construct an injection of free modules (γ, α): $G \to \bar{F} + F$ ($\operatorname{im}(\gamma, \alpha)$ is a "subkernel" as in (2.4)), where $\operatorname{im}(\gamma, \alpha)$ is self-annihilating under the canonical hyperbolic form on $\bar{F} + F$, $G \cong A^n \cong F$, and $\operatorname{cok}\alpha \cong M$, for $\alpha: G \to F$. To do this choose an epimorphism $k: F \to M$, G, and α so that

$$G \xrightarrow{\alpha} F \xrightarrow{k} M \longrightarrow 0$$

is exact and G and F are free of the same rank. Then $\ker\alpha + F \cong G + M$ so that $\ker\alpha \cong M$. Using k we obtain an injection $\bar{k}$: $\overline{M} \to \bar{F}$ and using the isomorphism $M \cong \overline{M}$ we define γ: $G \to \bar{F}$ to map $M = \ker\alpha \subseteq G$ isomorphically to $\overline{M} \subseteq \bar{F}$ and map some complement of M in G to zero. The reader may easily check that the map (γ, α) so constructed satisfies the required conditions.

To finish the proof of (3 1) we will construct the isomorphism $\tilde{f}$. To do this we will construct a commutative ladder with exact rows:

$$\begin{tikzcd} Z_2 \arrow[r, "i"] \arrow[d, "="'] & W_1^{\lambda}(A) \arrow[r, two heads, "j"] \arrow[d, "f", "\cong"'] & L_{\lambda}(A) \arrow[d, "\tilde{f}"] \\ Z_2 \arrow[r, "p"] & \Phi^{(-\lambda)}(A) \arrow[r, two heads, "q"] & \tilde{\Phi}^{(-\lambda)}(A). \end{tikzcd} \tag{3.11}$$

The map i sends the generator to w_1 (2.5(i)), this is justified because (3.5a) shows w_1 has order 2. The map j is the cokernel of i by definition of $L_\lambda(A)$ (2.5). The map p is induced by sending the generator to $[A]$ in $\hat{H}^0(Z_2\,;\,K_0(A))$. The definition of f then shows the left-hand square commutes. The map q is induced by $K_0(A) \to \tilde{K}_0(A)$. Finally, a diagram chase shows there is a unique isomorphism $\tilde{f}$ making the right-hand square commute This completes the proof of (3 1).

(3.12) COROLLARY. *If $\sigma \in U_{2n}^{\lambda}(A)$ vanishes in $W_1^{\lambda}(A)$,*

$$\sigma = X_{-}(-\tau)\, w_n X_{-}(\lambda\alpha B^{-1})\, H(B) X_{+}(B^{-1}C),$$

where B and C may be determined by the form supported on $\ker\gamma$.

Proof. The decomposition claimed comes from (3.8) and the identities in [2, II.4]. Since $B = \gamma + \tau\alpha$ and $C = \delta + \tau\beta$, where τ is constructed in Step 3 of (3.1) using the form on $\ker\gamma$, one may (in theory, if not in practice) write out σ explicitly.

Remark. By [13] one may determine the extent of uniqueness in this decomposition.

Before concluding this section we will indicate how some well-known "reduction theorems" may be used to extend (3.1). Let A be a ring and $J \subseteq A$ a two-sided ideal such that A is complete in the J-adic topology and $J = \bar{J}$. Then according to [15, Lemma 5] the map $A \to A/J$ induces an isomorphism $W_1^{\lambda}(A) \cong W_1^{\lambda}(A/J)$ (Our $W_1^{\lambda}(A)$ is $L_\lambda^K(A)$ in the notation of [15].)

Consider the composition (also called f)

$$W_1^{\lambda}(A) \overset{\cong}{\to} W_1^{\lambda}(A/J) \to \Phi^{(-\lambda)}(A/J). \tag{3.13}$$

If A/J is semisimple, then the map is an isomorphism If M is an A-module and $M^\wedge$ denotes M/J, then since $\operatorname{cok}(\alpha^\wedge) \cong (\operatorname{cok}\alpha)^\wedge$ the composition f of (3 13) is given by $\sigma \to (\operatorname{cok}\alpha)^\wedge$, where as usual $\sigma = \left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right)$.

(3.14) THEOREM. *Suppose A is a J-adically complete ring where $J =$* Rad A. *Then*

$$f\colon W_1^{\lambda}(A) \cong \Phi^{(-\lambda)}(A/J)$$

and

$$\tilde{f} \cdot L_\lambda(A) \cong \tilde{\Phi}^{(-\lambda)}(A/J),$$

where if $\sigma = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ *then* $f(\sigma) = (\operatorname{cok} \alpha)^\wedge = \operatorname{cok} \alpha/(\operatorname{cok} \alpha) \cdot J$.

(3.15) *Remark.* (i) Theorem (3.14) applies for example to R_p, the completion of a ring R of algebraic integers with respect to a nonarchimedean prime p, and $J = pR_p$; or to $Z_p\pi$, where π is a finite p-group and J is the augmentation ideal.

(ii) Although one has the above-mentioned reduction theory we point out that if A is any semilocal Artin ring, Steps 1 and 2 of (3.1) show it is possible to construct a homomorphism from $W_1{}^\lambda(A)$ to a group which is identical with $\Phi^{(-\lambda)}(A)$, except that $K_0(A)$ is replaced by $G_0(A)$ (see [1]) in $\hat{H}^0(Z_2; K_0(A))$. The reason for this is that cok α need not be projective. If A is not Artin, e.g., $A = R_p$ as in (i), then a more refined invariant is needed.

4. Some Calculations

Theorem (3.1) yields and extends some results of [2, 4, 16] proved by different methods.

(4.1) Proposition. *Let A be a division ring with involution. Then*

$$L_\lambda(A) = 0$$

and

$$\begin{aligned} W_1{}^\lambda(A) &= Z_2 \qquad \textit{if the involution is trivial and } \lambda = 1; \\ &= 0 \qquad \textit{otherwise.} \end{aligned}$$

The nonzero representative is $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in U_2{}^1(A)$.

Proof. $\tilde{K}_0(A) = 0$ so $\tilde{\Phi}^{(-\lambda)}(A) = 0$; by (3.1), $L_\lambda(A) = 0$. Similarly, since $K_0(A) = Z$, $\Phi^{(-\lambda)}(A)$ is most Z_2 and is represented by the free A-module of rank 1, A. Since $f(w_1) = [A]$, the last statement of the proposition follows. Hence, we must show there is no nonsingular $(-\lambda)$-hermitian form $\varphi: A \times A \to A$ if and only if $\lambda = 1$ and the involution is trivial. The construction of φ amounts to finding $a \in A$ such that $a = b - \lambda\bar{b}$, for some $b \in A$. It is easy to verify this is impossible only when the involution is trivial and $\lambda = 1$.

If A is a simple algebra with involution, then since $K_0(A) = Z$, we have $\tilde{K}_0(A) = Z_n$, where $A = M_n(D)$, D a division ring Hence,

$$\begin{aligned} \hat{H}^0(Z_2, \tilde{K}_0(A)) &= Z_2 \qquad n \text{ even}; \\ &= 0 \qquad n \text{ odd} \end{aligned}$$

while $\hat{H}^0(Z_2; K_0(A)) = Z_2$. By (3 1), $W_1^\lambda(A) = 0 = L_\lambda(A)$ if the unique irreducible A-module supports a form and $L_\lambda(A) = 0$ if n is odd in any case; otherwise (4 3) $W_1^\lambda(A) = Z_2 = L_\lambda(A)$. By [2, I, Sect. 8], if $A = M_n(D)$, D a division ring, then there is induced an involution σ: $D \to D$ such that σ agrees with the given involution on the center of D. Let "^" denote σ-conjugate transpose, $(d_{ij})^\wedge = (\sigma d_{ji})$, (d_{ij}) an $(n \times n)$-matrix in $M_n(D)$. By the Skolem–Noether theorem there is a unit $h \in A$ such that $\bar{a} = ha^\wedge h^{-1}$, for all $a \in A$. The following result is contained in [2, I.8 1.11] (and was proved independently by the author).

(4 2) Proposition. *Let A be a simple algebra with involution. Then the irreducible A-module supports a $(-\lambda)$-hermitian form unless the involution $\sigma \cdot D \to D$ (described above) is trivial, so that D is a field and either*

(a) $\lambda = 1$, $A = M_n(D)$, *and* $\bar{a} = ha^\wedge h^{-1}$, *where* $h = \text{diag}(d_1, \ldots, d_n)$ *and* $\sigma d_i = d_i$, *for all* i; *or*

(b) $\lambda = -1$, $A = M_{2n}(D)$, *and* $\bar{a} = ha^\wedge h^{-1}$, *where* $h^\wedge = -h$.

(4.3) Corollary. *Let A be a simple algebra with involution. Then*

$$\begin{aligned} W_1^\lambda(A) &= Z_2 && \text{if } \lambda \text{ and } A \text{ are as in (4.2)(a) or (b);} \\ &= 0 && \text{otherwise.} \end{aligned}$$

and

$$\begin{aligned} L_\lambda(A) &= Z_2 && \text{if } \lambda \text{ and } A \text{ are as in (4.2)(a) with } n \text{ even; or as in (4.2)(b);} \\ &= 0 && \text{otherwise.} \end{aligned}$$

In all cases, the nonzero representative is given by $\sigma_e = \begin{pmatrix} 1-e & e \\ \lambda\bar{e} & 1-\bar{e} \end{pmatrix} \in U_2^\lambda(A)$, *where* $e \in A$ *and* eA *is an irreducible right A-module.*

Proof. The "otherwise" statements follow from (4 2) as we noted above. For the rest we need to show that if M is a simple A-module in cases (4.2)(a) or (b), then $[M] \neq 0$ in $\Phi^{(-\lambda)}(A)$. If not, then the reasoning leading to (3.6a) and (3.6b) shows that $M + A^{2k}$ supports a $(-\lambda)$-form,

$$h: (M + A^{2k}) \times (M + A^{2k}) \to A$$

for some k. Note that $M + A^{2k}$ is isomorphic to the sum $\sum_i e_i A$ of an odd number of copies of eA, an irreducible A-module. If, say, $h(e_1, e_1) = 0$, then $h(e_1 A, e_1 A) \equiv 0$ and we may find a submodule N of $M + A^p$, isomorphic to $e_1 A$ such that $h \mid (e_1 A + N) \times (e_1 A + N)$ is the hyperbolic pairing. Taking the orthogonal complement we find a form on an odd number of copies of eA. Continuing in this way, either we find that $h(e, e) \neq 0$, in which case we are done, or we reduce the number of copies of eA to one, which also gives us the required contradiction. Since we may take $e^2 = e$, it follows that $\sigma_e \in U_2{}^\lambda(A)$. By definition $\operatorname{cok}(m_{1-e}) = eA$, where m_a denotes left multiplication by $a \in A$. This completes the proof

Now recall the following definition from [2, IV.2.1].

(4.4) Definition. $\Lambda^\lambda = \{a - \lambda\bar{a} \mid a \in A\}$ is said to be *ample in* A if, for each $a, b \in A$, there exists $r \in \Lambda^\lambda$ such that $A(a + rb) = Aa + Ab$.

(4 5) Proposition. *Let A be a simple ring with involution Then Λ^λ is ample in A if and only if $U_2{}^\lambda(A)$ is generated by elements of the form $X_+(\rho)$, $X_-(\tau)$, and $H(a)$, where $\rho, \tau \in \Lambda^\lambda$ and a is a unit in A*

Proof. By [2, IV.2.1], if Λ^λ is ample then $U_2{}^\lambda(A) = \langle X_+(\rho), X_-(\tau), H(a)\rangle$. Conversely, if $U_2{}^\lambda(A) = \langle X_+(\rho), X_-(\tau), H(a)\rangle$, then by (3.1), each $\sigma \in U_2{}^\lambda(A)$,

$$\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with entries A, must be such that $\operatorname{cok}(m_a : A \to A)$ stably supports a form, m_a denotes left multiplication by a. In particular, if

$$\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 - e & e \\ \lambda\bar{e} & 1 - \bar{e} \end{pmatrix},$$

where eA is an irreducible A-module, then by (3.1), (4.2), and (4.3), A and λ can be any algebra and symmetry except those of (4 2)(a) or (b). By [2, IV.2 3], Λ^λ is ample in A.

As a final example, we consider certain semisimple rings A with more than one simple factor. Since these are results with mainly geometric application (to be explained in Sect. 5) we will work with L_λ instead of $W_1{}^\lambda$. Let π be a group, R a ring, and let the group ring $R\pi$ be endowed with the involution induced by $g \to g^{-1}$, $g \in \pi$.

(4.6) Theorem. *Let π be a finite group of odd order Then for $\lambda = \pm 1$,*

(a) $L_\lambda(Q\pi) = 0$,

(b) $L_\lambda(F_2\pi) = 0$.

Proof. Let K denote either Q or F_2, the field of 2-elements. We have $K\pi \cong K \times \prod_i M_{n_i}(D_i)$, where the factor K on the right-hand side corresponds to the trivial one-dimensional K-representation of π, and where the D_i are division algebras over K. The involution on $K\pi$, being of order 2, permutes some pairs of simple factors of $K\pi$, $M_n(D) \times M_n{}'(D)$, and fixes the rest. Hence, as a ring with involution

$$K\pi \cong K \times \Big(\prod_i M_{n_i}(D_i) \times M'_{n_i}(D_i)\Big) \times \prod_j M_{n_j}(D_j) \tag{4.7}$$

Obviously, modules over the permuted pairs in (4.7) support (hyperbolic) $(-\lambda)$-forms. Hence, since $[K\pi] = 0$ in $\tilde{\Phi}^{(-\lambda)}(K\pi)$, it suffices to show that each irreducible $M_{n_j}(D_j)$-module supports a $(-\lambda)$-form, where $M_{n_j}(D_j)$ in (4.7) is taken to itself by the involution on $K\pi$. To do this, we will show that the involution induced on $Z(D)$ (= the center of D) is nontrivial and then apply (4.2).

In case(b), $K = F_2$, this is Theorem 1.6 of [11]. In case (a), $K = Q$, we note that since $M_n(D) \otimes_Q R \cong M_n(D \otimes_Q R)$, so that $Z(M_n(D) \otimes_Q R) \cong Z(D \otimes_Q R)$, it suffices to see that the involutions on the centers of the fixed components of $R\pi$ are nontrivial. Since $|\pi|$ is odd, a theorem of Burnside [14, p. 124] states that each irreducible nontrivial complex character of π takes at least one complex nonreal value By the analysis of [14, p. 122] $R\pi \cong R \times \prod_i M_{n_i}(C)$, where the involution on each copy of C is complex conjugation. This completes the proof.

In one case, the L-group is zero for all finite π.

(4 8) THEOREM $L_{-1}(Q\pi) = 0$ *for* π *finite.*

Proof We will construct a symmetric hermitian form $\eta \cdot N \times N \to Q\pi$, where N is a $Q\pi$-irreducible module. First let φ: $N \times N \to Q$ be any positive definite symmetric form on the Q-module N. Then ψ: $N \times N \to Q$, given by

$$\psi(m, n) = \sum_{g \in \pi} \varphi(mg, ng), \qquad m, n \in N$$

is easily seen to be nonsingular symmetric and to satisfy $\psi(mk, nk) = \psi(m, n)$ for all $m, n \in N$; $k \in \pi$. Finally, set

$$\eta(m, n) = \sum_{g \in \pi} \psi(m, ng^{-1})\, g \in Q\pi.$$

Then η: $N \times N \to Q\pi$ can be shown to satisfy (2.2) (1–5) for $\lambda = +1$.

(4.9) COROLLARY. *The factor of type* (b) *in* (4.2) *cannot occur in* $Q\pi$, π *finite.*

5. CONNECTIONS WITH A ROTHENBERG SEQUENCE, R. LEE'S SEMICHARACTERISTIC, AND SURGERY THEORY

This section contains some informal remarks about (3.1).

(A) In [12] (4 3) there is a "Rothenberg sequence" (conjectured in [17]), the relevant part of which for us is

$$W_0^{(-\lambda)}(A) \xrightarrow{F} \hat{H}^0(Z_2\,; K_0(A)) \xrightarrow{E} W_1^{\lambda}(A) \xrightarrow{D} U_1^{\lambda}(A) \longrightarrow \hat{H}^1(Z_2\,; K_0(A)). \quad (5.1)$$

Here F is the forgetful functor (3.0) whose cokernel we have denoted $\Phi^{(-\lambda)}(A)$; $U_1^{\lambda}(A)$ (denoted $U_\lambda(A)$ in [12]) is a group of equivalence classes of *projective* subkernels G of $\bar{F}+F$ in the manner of (2.4) above; and the map E may be shown to be given in Step 4 of (3 1). So our main theorem (3.1) implies that $D=0$, or that $U_1^\lambda(A)$ is detected by its image in $\hat{H}^1(Z_2\,;K_0(A))$, given by the class of G, the subkernel of $\bar{F}+F$.

(B) (The references for unexplained terminology in this section are [3, 17].) Given a degree one normal map of compact oriented $(2k+1)$-manifolds, $f\colon M\to N$, with $\pi_1 N=\pi$, one may assume (modulo an equivalence relation called normal cobordism of the map f) that the map induced in homology by f has nonzero kernels in dimensions k and $k+1$, $K_k(f)$ and $K_{k+1}(f)$. These are $R\pi$-modules in a natural way for any ring R and Lee (essentially) showed that when $R\pi$ is semisimple, the class of $K_k(f)$ in $\tilde{\Phi}^{(-1)^{k+1}}(R\pi)$ is a normal cobordism invariant of f [8]. In [17, Chap. 6] it is further shown that, in favorable cases, $L_{(-1)^k}(R\pi)$ is exactly the set of normal cobordism classes. It is easy to show that in the correspondences

$$L_{(-1)^k}(R\pi)\leftrightarrow\left\{\begin{matrix}\text{normal cobordism classes}\\ \text{of normal maps}\end{matrix}\right\}\leftrightarrow\tilde{\Phi}^{(-1)^{k+1}}(R\pi),$$

$\sigma=\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$ on the left goes to cok α on the right. This geometric example is the motivation for (3 1), and (3.1) was first proved for $A=Q\pi$ and the L_λ-groups in [9] and, independently, [6].

(C) The main theorem of [11] is now an easy consequence of our discussion in (B) above and Theorem (4.6)(b) Also our computations in Section 4 reduce the computation of the terms $W_1^\lambda(\bar{A})$ and $L_\lambda(\bar{A})$ to questions about the simple factors of $\bar{A}$, for many rings A appearing in the localization sequence for quadratic forms in [10]. This will be used in determining the L- and W-groups of the localization sequence.

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